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NUMERICAL SOLUTION OF TENTH ORDER BOUNDARY VALUE PROBLEMS BY GALERKIN METHOD WITH SEXTIC B-SPLINES

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ABSTRACT

A finite element method involving Galerkin method with sextic B-splines as basis functions has been developed to solve a general tenth order boundary value problem. The basis functions are redefined into a new set of basis functions which vanish on the boundary where the Dirichlet, Neumann, second order derivative, third order derivative and fourth order derivative types of boundary conditions are prescribed. The proposed method was applied to solve several examples of tenth order linear and nonlinear boundary value problems. The solution of a nonlinear boundary value problem has been obtained as the limit of a sequence of solution of linear boundary value problems generated by quasilinearization technique. The obtained numerical results are compared with the exact solutions available in the literature.

KEYWORDS: Absolute Error, Basis Function, Galerkin Method, Sextic B-Spline, Tenth Order Boundary Value Problem

INTRODUCTION

In this paper, we consider a general tenth order linear boundary value problem given by

$$\begin{aligned}&a_0(x)y^{(10)}(x)+a_1(x)y^{(9)}(x)+a_2(x)y^{(8)}(x)+a_3(x)y^{(7)}(x)+a_4(x)y^{(6)}(x)+a_5(x)y^{(5)}(x)+a_6(x)y^{(4)}(x)\\&+a_7(x)y'''(x)+a_8(x)y''(x)+a_9(x)y'(x)+a_{10}(x)y(x)=b(x),\quad c<x<d\end{aligned}\tag{1}$$

Subject to boundary conditions

$$y(c)=A_0, y(d)=C_0, y'(c)=A_1, y'(d)=C_1, y''(c)=A_2, y''(c)=C_2, y'''(c)=A_3, y'''(c)=C_3, y^{(4)}(c)=A_4, y^{(4)}(d)=C_4.\tag{2}$$

Where A_0, C_0, A_1, C_1, A_2, C_2, A_3, C_3, A_4, C_4 are finite real constants and $a_0(x)$, $a_1(x)$, $a_2(x)$, $a_3(x)$, $a_4(x)$, $a_5(x)$, $a_6(x)$, $a_7(x)$, $a_8(x)$, $a_9(x)$, $a_{10}(x)$, $b(x)$ are all continuous functions defined on the interval [c, d].

Generally, this type of tenth order boundary value problem arises in the study of hydrodynamics and hydro magnetic stability, mathematical modeling of the viscoelastic flows and other areas of applied mathematics, physics, engineering sciences. When an infinite horizontal layer of fluid is heated from below and is under the action of rotation, instability sets in. When this instability is an ordinary convection, the ordinary differential equation is of sixth order. When the instability sets in as over stability, it is modeled by an eighth order ordinary differential equation. Suppose, now that a uniform magnetic field is also applied across the fluid in the same direction as gravity. When instability sets now as ordinary convection, it is modeled by tenth order boundary value problem [1].

The existence and uniqueness of solutions of these problems have been discussed by Agarwal [2]. The boundary value problems of higher order differential equations have been investigated due to their mathematical importance and the potential for applications in diversified applied sciences. Solving these type of boundary value problems analytically is very difficult and analytical solutions are available in very rare cases. Very few authors have attempted the numerical solution of tenth order boundary value problems. Some of the numerical methods have been developed overs the years to approximate the solution for these type of boundary value problems. Twizell *et al*. [3] developed numerical methods for eight, tenth, twelfth order eigen value problems arising in thermal instability. Siddiqi and Twizell [4] developed the solution of special case of tenth order boundary value problems using tenth degree splines. Ghazala Akram and Siddiqi [5, 6] presented the solution of special case of tenth order boundary value problems using an eleventh degree polynomial and non-polynomial splines. Scott and Watts [7] applied a combination of superposition and orthonormalization to solve a linear boundary value problem. Scott and Watts [8] described several computer codes that were developed using superposition and orthonormalization technique and invariant imbedding. Rashidinia *et al*. [9] presented the solution of special case of tenth order boundary value problems using a eleventh degree non-polynomial splines technique. Dijidejeli and Twizell [10] derived numerical method for special case of boundary value problems of order 2m. Laminni *et al.* [11] developed and analyzed numerical method for approximating the solution of linear boundary value problems. Ramadan *et al.* [12] have applied non-polynomial spline functions for approximating the solutions of $(2\mu)^{th}$ order two point boundary value problems. Erturk and Momani [13] applied Differential transform method to construct the solution for tenth order boundary value problems. Wazwaz [14] presented a modified Adomain decomposition method for tenth and twelfth order boundary value problems. Farajeyan and Maleki [15] have applied a eleventh degree nonpolynomial off step spline technique. Noor *et al.* [16] developed a reliable algorithm for solving special case of tenth order boundary value problems. Gang and Li [17] presented the solution of special case of tenth order boundary value problems by using Variational iteration method. Barai *et al.* [18] applied Homotopy perturbation method for solving tenth order boundary value problems. Mohyudin and Ahmet [19] have applied modified Variational iteration method for solving tenth and ninth order boundary value problems. Siddiqi *et al.* [20] presented the solution of special case of tenth order boundary value problems by using the Variational iteration method. Kasi Viswanadham and Showri Raju [21] developed a quintic B-spline Collocation method for solving a general tenth order boundary value problem. So far, tenth order boundary value problems have not been solved by using Galerkin method with sextic B-splines. This motivated us to solve a general tenth order boundary value problem by Galerkin method with sextic B-splines. The proposed method is tested on several linear and nonlinear boundary value problems. The solution to a nonlinear problem has been obtained as the limit of a sequence of solution of linear problems generated by the quasilinearization technique [22].

JUSTIFICATION FOR USING GALERKIN METHOD

For the few decades, the finite element method (FEM) has become very powerful, useful tool to solve the boundary value problems in the complex geometry. In FEM, the approximate solution can be written as a linear combination of basis functions which constitute a basis for the approximation space under consideration. FEM involves variational methods like Rayleigh Ritz, Galerkin, Petrov-Galerkin, Least Squares and Collocation etc.

In Galerkin method, the residual of approximation is made orthogonal to the basis functions. When one uses Galerkin method, a weak form of approximation solution for a given differential equation exists and is unique under appropriate conditions [23, 24] irrespective of properties of a given differential operator. Further, a weak solution also tends to a classical solution of given differential equation, provided sufficient attention is given to boundary conditions [25]. That means the basis functions should vanish on the boundary where the Dirichlet type of boundary conditions are prescribed. Hence in this paper we employed the use of Galerkin method with sextic B-splines as basis functions to approximate the solution of tenth order boundary value problems.

DESCRIPTION OF THE METHOD

Definition of Sextic B-Spline: The sextic B-splines are defined in [26-28]. The existence of sextic spline interpolate $s(x)$ to a function in a closed interval $[c, d]$ for spaced knots (need not be evenly spaced) of a partition $c = x_0 < x_1 < \ldots < x_{n-1} < x_n = d$ is established by constructing it. The construction of $s(x)$ is done with the help of the sextic B-splines. Introduce twelve additional knots x_{-6}, x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}, x_{n+5} and x_{n+6} in such a way that

$x_{-6}<x_{-5}<x_{-4}< x_{-3}<x_{-2}<x_{-1}<x_0$ and $x_n<x_{n+1}<x_{n+2}<x_{n+3}<x_{n+4}<x_{n+5}<x_{n+6}.$

Now the sextic B-splines $B_i(x)'s$ are defined by

$$B_i(x) = \begin{cases} \sum_{r=i-3}^{i+4} \dfrac{(x_r - x)_+^6}{\pi'(x_r)}, & x \in [x_{i-3}, x_{i+4}] \\ 0, & otherwise \end{cases}$$

where $(x_r - x)_+^6 = \begin{cases} (x_r - x)^6, & if\ x_r \geq x \\ 0, & if\ x_r \leq x \end{cases}$

and $\pi(x) = \prod_{r=i-3}^{i+4} (x - x_r)$

where $\{B_{-3}(x), B_{-2}(x), B_{-1}(x), B_0(x), B_1(x),\ldots,B_{n-1}(x), B_n(x), B_{n+1}(x), B_{n+2}(x)\}$ forms a basis for the space $S_6(\pi)$ of sextic polynomial splines. Schoenberg [28] has proved that sextic B-splines are the unique nonzero splines of smallest compact support with the knots at

$x_{-6}<x_{-5}<x_{-4}<x_{-3}<x_{-2}<x_{-1}<x_0<x_1<\ldots<x_{n-1}<x_n<x_{n+1}<x_{n+2}<x_{n+3}<x_{n+4}<x_{n+5}<x_{n+6}.$

To solve the boundary value problem (1) and (2) by the Galerkin method with sextic B-splines as basis functions, we define the approximation for $y(x)$ as

$$y(x) = \sum_{j=-3}^{n+2} \alpha_j B_j(x) \tag{3}$$

where α_j's are the nodal parameters to be determined. In Galerkin method, the basis functions should vanish on the boundary where the Dirichlet type of boundary conditions are specified. In the set of sextic B-splines $\{B_{-3}(x), B_{-2}(x), B_{-1}(x), B_0(x),\ldots, B_n(x), B_{n+1}(x), B_{n+2}(x)\}$, the basis functions $B_{-3}(x)$, $B_{-2}(x)$, $B_{-1}(x)$, $B_0(x)$, $B_1(x)$, $B_2(x)$, $B_{n-3}(x)$, $B_{n-2}(x)$, $B_{n-1}(x)$, $B_n(x)$, $B_{n+1}(x)$ and $B_{n+2}(x)$ do not vanish at one of the boundary points. So, there is a necessity of redefining the basis functions into a new set of basis functions which vanish on the boundary where the Dirichlet type of boundary conditions are specified. Since, we are approximating the tenth order boundary value problem by sextic B-splines polynomial, we redefine the basis functions into a new set of basis functions which vanish on the boundary where the Dirichlet, Neumann, second order derivative, third order derivative and fourth order derivative types of boundary conditions are prescribed. The procedure for redefining of the basis functions is as follows.

Using the definition of sextic B-splines and the Dirichlet boundary conditions of (2), we get the approximate solution at the boundary points as

$$A_0 = y(c) = y(x_0) = \alpha_{-3}B_{-3}(x_0) + \alpha_{-2}B_{-2}(x_0) + \alpha_{-1}B_{-1}(x_0) + \alpha_0 B_0(x_0) + \alpha_1 B_1(x_0) + \alpha_2 B_2(x_0) \tag{4}$$

$$C_0 = y(d) = y(x_n) = \alpha_{n-3}B_{n-3}(x_n) + \alpha_{n-2}B_{n-2}(x_n) + \alpha_{n-1}B_{n-1}(x_n) + \alpha_n B_n(x_n) + \alpha_{n+1}B_{n+1}(x_n) + \alpha_{n+2}B_{n+2}(x_n) \tag{5}$$

Eliminating α_{-3} and α_{n+2} from the equations (3), (4) and (5), we get

$$y(x) = w_1(x) + \sum_{j=-2}^{n+1} \alpha_j P_j(x) \tag{6}$$

where $w_1(x) = \dfrac{A_0}{B_{-3}(x_0)} B_{-3}(x) + \dfrac{C_0}{B_{n+2}(x_n)} B_{n+2}(x)$ (7)

and
$$P_j(x) = \begin{cases} B_j(x) - \dfrac{B_j(x_0)}{B_{-3}(x_0)} B_{-3}(x), & j = -2, -1, 0, 1, 2 \\ B_j(x), & j = 3, \ldots, n-4 \\ B_j(x) - \dfrac{B_j(x_n)}{B_{n+2}(x_n)} B_{n+2}(x), & j = n-3, n-2, n-1, n, n+1. \end{cases} \tag{8}$$

Using the Neumann boundary conditions of (2) to the approximate solution *y(x)* given by (6), we get

$$A_1 = y'(c) = y'(x_0) = w_1'(x_0) + \alpha_{-2}P_{-2}'(x_0) + \alpha_{-1}P_{-1}'(x_0) + \alpha_0 P_0'(x_0) + \alpha_1 P_1'(x_0) + \alpha_2 P_2'(x_0) \tag{9}$$

$$C_1 = y'(d) = y'(x_n) = w_1'(x_n) + \alpha_{n-3}P_{n-3}'(x_n) + \alpha_{n-2}P_{n-2}'(x_n) + \alpha_{n-1}P_{n-1}'(x_n) + \alpha_n P_n'(x_n) + \alpha_{n+1}P_{n+1}'(x_n) \tag{10}$$

Eliminating α_{-2} and α_{n+1} from the equations (6), (9) and (10), we get the approximation for *y(x)* as

$$y(x) = w_2(x) + \sum_{j=-1}^{n} \alpha_j Q_j(x) \tag{11}$$

where $w_2(x) = w_1(x) + \dfrac{A_1 - w_1'(x_0)}{P_{-2}'(x_0)} P_{-2}(x) + \dfrac{C_1 - w_1'(x_n)}{P_{n+1}'(x_n)} P_{n+1}(x)$ (12)

and
$$Q_j(x) = \begin{cases} P_j(x) - \dfrac{P_j'(x_0)}{P_{-2}'(x_0)} P_{-2}(x), & j = -1, 0, 1, 2 \\ P_j(x), & j = 3, \ldots, n-4 \\ P_j(x) - \dfrac{P_j'(x_n)}{P_{n+1}'(x_n)} P_{n+1}(x), & j = n-3, n-2, n-1, n. \end{cases} \tag{13}$$

Using the second order derivative boundary conditions of (2) to the approximate solution *y(x)* given by (11), we get

$$A_2 = y''(c) = y''(x_0) = w_2''(x_0) + \alpha_{-1}Q_{-1}''(x_0) + \alpha_0 Q_0''(x_0) + \alpha_1 Q_1''(x_0) + \alpha_2 Q_2''(x_0) \tag{14}$$

$$C_2 = y''(d) = y''(x_n) = w_2''(x_n) + \alpha_{n-3}Q_{n-3}''(x_n) + \alpha_{n-2}Q_{n-2}''(x_n) + \alpha_{n-1}Q_{n-1}''(x_n) + \alpha_n Q_n''(x_n) \tag{15}$$

Eliminating α_{-1} and α_n from the approximations (11), (14) and (15), we get the approximation for *y(x)* as

$$y(x) = w_3(x) + \sum_{j=0}^{n-1} \alpha_j R_j(x) \tag{16}$$

where $w_3(x) = w_2(x) + \dfrac{A_2 - w_2''(x_0)}{Q_{-1}''(x_0)} Q_{-1}(x) + \dfrac{C_2 - w_2''(x_n)}{Q_n''(x_n)} Q_n(x)$ (17)

and

$$R_j(x) = \begin{cases} Q_j(x) - \dfrac{Q_j''(x_0)}{Q_{-1}''(x_0)} Q_{-1}(x), & j = 0,1,2 \\ Q_j(x), & j = 3,\ldots,n-4 \\ Q_j(x) - \dfrac{Q_j''(x_n)}{Q_n''(x_n)} Q_n(x), & j = n-3, n-2, n-1. \end{cases} \tag{18}$$

Using the third order derivative boundary conditions of (2) to the approximate solution *y(x)* given by (16), we get

$$A_3 = y'''(c) = y'''(x_0) = w_3'''(x_0) + \alpha_0 R_0'''(x_0) + \alpha_1 R_1'''(x_0) + \alpha_2 R_2'''(x_0) \tag{19}$$

$$C_3 = y'''(d) = y'''(x_n) = w_3'''(x_n) + \alpha_{n-3}R_{n-3}'''(x_n) + \alpha_{n-2}R_{n-2}'''(x_n) + \alpha_{n-1}R_{n-1}'''(x_n) \tag{20}$$

Eliminating α_0 and α_{n-1} from the equations (16), (19) and (20), we get the approximation for $y(x)$ as

$$y(x) = w_4(x) + \sum_{j=1}^{n-2} \alpha_j S_j(x) \tag{21}$$

where

$$S_j(x) = \begin{cases} R_j(x) - \dfrac{R_j'''(x_0)}{R_0'''(x_0)} R_0(x), & j = 1,2 \\ R_j(x), & j = 3,\ldots,n-4 \\ R_j(x) - \dfrac{R_j'''(x_n)}{R_{n-1}'''(x_n)} R_{n-1}(x), & j = n-3, n-2. \end{cases} \tag{22}$$

and $w_4(x) = w_3(x) + \dfrac{A_3 - w_3'''(x_0)}{R_0'''(x_0)} R_0(x) + \dfrac{C_3 - w_3'''(x_n)}{R_{n-1}'''(x_n)} R_{n-1}(x)$ (23)

Using the fourth order derivative boundary conditions of (2) to the approximate solution *y(x)* given by (21), we get

$$A_4 = y^{(4)}(c) = y^{(4)}(x_0) = w_4^{(4)}(x_0) + \alpha_1 S_1^{(4)}(x_0) + \alpha_2 S_2^{(4)}(x_0) \tag{24}$$

$$C_4 = y^{(4)}(d) = y^{(4)}(x_n) = w_4^{(4)}(x_n) + \alpha_{n-3} S_{n-3}^{(4)}(x_n) + \alpha_{n-2} S_{n-2}^{(4)}(x_n) \tag{25}$$

Eliminating α_1 and α_{n-2} from the equations (21), (24) and (25), we get the approximation for $y(x)$ as

$$y(x) = w(x) + \sum_{j=2}^{n-3} \alpha_j \tilde{B}_j(x) \tag{26}$$

where $w(x) = w_4(x) + \dfrac{A_4 - w_4^{(4)}(x_0)}{S_1^{(4)}(x_0)} S_1(x) + \dfrac{C_4 - w_4^{(4)}(x_n)}{S_{n-2}^{(4)}(x_n)} S_{n-2}(x)$ (27)

and
$$\tilde{B}_j(x) = \begin{cases} S_j(x) - \dfrac{S_j^{(4)}(x_0)}{S_1^{(4)}(x_0)} S_1(x), & j = 2 \\ S_j(x), & j = 3, \ldots, n-4 \\ S_j(x) - \dfrac{S_j^{(4)}(x_n)}{S_{n-2}^{(4)}(x_n)} S_{n-2}(x), & j = n-3. \end{cases} \tag{28}$$

Now the new set of basis functions for the approximation *y(x)* is $\{\tilde{B}_j(x),\, j = 2, \ldots, n-3\}$. Applying the Galerkin method to (1) with a new set of basis functions, we get

$$\begin{aligned} &\int_{x_0}^{x_n} \big[a_0(x) y^{(10)}(x) + a_1(x) y^{(9)}(x) + a_2(x) y^{(8)}(x) + a_3(x) y^{(7)}(x) + a_4(x) y^{(6)}(x) + a_5(x) y^{(5)}(x) + a_6(x) y^{(4)}(x) + a_7(x) y'''(x) \\ &+ a_8(x) y''(x) + a_9(x) y'(x) + a_{10}(x) y(x) \big] \tilde{B}_i(x)\, dx = \int_{x_0}^{x_n} b(x) \tilde{B}_i(x) dx \quad \text{for} \quad i = 2, 3, \ldots, n-3. \end{aligned} \tag{29}$$

Integrating by parts the first five terms on the left hand side of (29) and after applying the boundary conditions prescribed in (2), we get

$$\int_{x_0}^{x_n} a_0(x) \tilde{B}_i(x) y^{(10)}(x) dx = -\int_{x_0}^{x_n} \frac{d^5}{dx^5} \big[a_0(x) \tilde{B}_i(x) \big] y^{(5)}(x) dx \tag{30}$$

$$\int_{x_0}^{x_n} a_1(x) \tilde{B}_i(x) y^{(9)}(x) dx = \int_{x_0}^{x_n} \frac{d^4}{dx^4} \big[a_1(x) \tilde{B}_i(x) \big] y^{(5)}(x) dx \tag{31}$$

$$\int_{x_0}^{x_n} a_2(x) \tilde{B}_i(x) y^{(8)}(x) dx = -\int_{x_0}^{x_n} \frac{d^3}{dx^3} \big[a_2(x) \tilde{B}_i(x) \big] y^{(5)}(x) dx \tag{32}$$

$$\int_{x_0}^{x_n} a_3(x) \tilde{B}_i(x) y^{(7)}(x) dx = \int_{x_0}^{x_n} \frac{d^2}{dx^2} \big[a_3(x) \tilde{B}_i(x) \big] y^{(5)}(x) dx \tag{33}$$

$$\int_{x_0}^{x_n} a_4(x) \tilde{B}_i(x) y^{(6)}(x) dx = -\int_{x_0}^{x_n} \frac{d}{dx} \big[a_4(x) \tilde{B}_i(x) \big] y^{(5)}(x) dx \tag{34}$$

Substituting (30) to (34) in (29) and using the approximation for $y(x)$ given in (26), and after rearranging the terms for resulting equations, we get a system of equations in the matrix form as

$$\mathbf{A}\alpha = \mathbf{B} \tag{35}$$

where $\mathbf{A} = [a_{ij}]$;

$$\begin{aligned} a_{ij} = \int_{x_0}^{x_n} \{[& -\frac{d^5}{dx^5}(a_0(x)\tilde{B}_i(x)) + \frac{d^4}{dx^4}(a_1(x)\tilde{B}_i(x)) - \frac{d^3}{dx^3}(a_2(x)\tilde{B}_i(x)) + \frac{d^2}{dx^2}(a_3(x)\tilde{B}_i(x)) - \frac{d}{dx}(a_4(x)\tilde{B}_i(x)) \\ & + a_5(x)\tilde{B}_i(x)]\tilde{B}_j^{(5)}(x) + a_6(x)\tilde{B}_i(x)\tilde{B}_j^{(4)}(x) + a_7(x)\tilde{B}_i(x)\tilde{B}_j'''(x) + a_8(x)\tilde{B}_i(x)\tilde{B}_j''(x) + a_9(x)\tilde{B}_i(x)\tilde{B}_j'(x) \\ & + a_{10}(x)\tilde{B}_i(x))\tilde{B}_j(x)\}dx \quad \text{for} \quad \mathrm{i} = 2,3,\ldots, \mathrm{n}-3;\ \mathrm{j} = 2,3,\ldots,\mathrm{n}-3. \end{aligned} \tag{36}$$

$\mathbf{B} = [b_i]$;

$$\begin{aligned} b_i = \int_{x_0}^{x_n} \{ & b(x)\tilde{B}_i(x) + [\frac{d^5}{dx^5}(a_0(x)\tilde{B}_i(x)) - \frac{d^4}{dx^4}(a_1(x)\tilde{B}_i(x)) + \frac{d^3}{dx^3}(a_2(x)\tilde{B}_i(x)) - \frac{d^2}{dx^2}(a_3(x)\tilde{B}_i(x)) \\ & + \frac{d}{dx}(a_4(x)\tilde{B}_i(x)) - a_5(x)\tilde{B}_i(x)]w^{(5)}(x) - a_6(x)\tilde{B}_i(x)w^{(4)}(x) - a_7(x)\tilde{B}_i(x)w'''(x) - a_8(x)\tilde{B}_i(x)w''(x) \\ & - a_9(x)\tilde{B}_i(x)w'(x) - a_{10}(x)\tilde{B}_i(x))w(x)\}dx \quad \text{for} \quad \mathrm{i} = 2,3,\ldots, \mathrm{n}-3. \end{aligned} \tag{37}$$

and $\alpha = [\alpha_2\ \alpha_3 \ldots \alpha_{n-3}]^T$.

PROCEDURE TO FIND A SOLUTION FOR NODAL PARAMETER

A typical integral element in the matrix $\mathbf{A}$ is

$$\sum_{m=0}^{n-1} I_m$$

where $I_m = \int_{x_m}^{x_{m+1}} r_i(x) r_j(x) Z(x) dx$ and $r_i(x), r_j(x)$ are the sextic B-spline basis functions or their derivatives. It may be noted that $I_m = 0$ if $(x_{i-3}, x_{i+4}) \cap (x_{j-3}, x_{j+4}) \cap (x_m, x_{m+1}) = \varnothing$. To evaluate each I_m, we employed 7-point Gauss-Legendre quadrature formula. Thus the stiffness matrix $\mathbf{A}$ is a thirteen diagonal band matrix. The nodal parameter vector α has been obtained from the system $\mathbf{A}\alpha = \mathbf{B}$ by using a band matrix solution package. We have used the FORTRAN-90 program to solve the boundary value problems (1)-(2) by the proposed method.

NUMERICAL RESULTS

To test the efficiency of the proposed method for solving the tenth order boundary value problems of the types (1) and (2), we considered three linear boundary value problems and three nonlinear boundary value problems. Numerical results for each problem are presented in tabular forms and compared with the exact solutions available in the literature.

Example 1: Consider the linear boundary value problem

$$y^{(10)} + 5y = 10\cos x + 4(x-1)\sin x, \qquad 0 < x < 1 \tag{38}$$

Subject to $y(0) = y(1) = 0,\ y'(0) = -1,\ y'(1) = \sin 1,\ \ y''(0) = 2,\ y''(1) = 2\cos 1,$

$y'''(0) = 1,\ y'''(1) = -3\sin 1,\ \ y^{(4)}(0) = -4,\ y^{(4)}(1) = -4\cos 1$.

The exact solution for the above problem is $y = (x-1)\sin x.$

The proposed method is tested on this problem where the domain [0, 1] is divided into 10 equal subintervals. The obtained numerical results for this problem are given in Table 1. The maximum absolute error obtained by the proposed method is $3.516674\text{x}10^{-6}$.

Example 2: Consider the linear boundary value problem

$$y^{(10)} - (x^2 - 2x)y = 10\cos x - (x-1)^3 \sin x, \qquad -1 \le x \le 1 \tag{39}$$

subject to

$y(-1) = 2\sin 1,\ y(1) = 0,\ y'(-1) = -2\cos 1 - \sin 1,\ y'(1) = \sin 1,\ y''(-1) = 2\cos 1 - 2\sin 1,\ y''(1) = 2\cos 1,$

$y'''(-1) = 2\cos 1 + 3\sin 1,\ y'''(1) = -3\sin 1,\ y^{(4)}(-1) = -4\cos 1 + 2\sin 1,\ y^{(4)}(1) = -4\cos 1.$

The exact solution for the above problem is $y = (x-1)\sin x.$

The proposed method is tested on this problem where the domain [-1, 1] is divided into 10 equal subintervals. The obtained numerical results for this problem are given in Table 2. The maximum absolute error obtained by the proposed method is $2.441257\text{x}10^{-5}$.

Table 1: Numerical Results for Example 1

x	Exact Solution	Absolute Error by Proposed Method
0.1	-8.985008E-02	1.788139E-07
0.2	-1.589355E-01	1.192093E-07
0.3	-2.068641E-01	2.875924E-06
0.4	-2.336510E-01	3.606081E-06
0.5	-2.397128E-01	2.443790E-06
0.6	-2.258570E-01	3.516674E-06
0.7	-1.932653E-01	2.250075E-06
0.8	-1.434712E-01	1.639128E-06
0.9	-7.833266E-02	2.145767E-06

Table 2: Numerical Results for Example 2

x	Exact Solution	Absolute Error by Proposed Method
-0.8	1.291241	9.536743E-07
-0.6	9.034280E-01	2.205372E-06
-0.4	5.451856E-01	8.761883E-06
-0.2	2.384032E-01	1.777709E-05
0.0	0.0000000000	2.441257E-05
0.2	-1.589355E-01	2.214313E-05

Table 2: Contd.,

0.4	-2.336510E-01	1.828372E-05
0.6	-2.258570E-01	1.232326E-05
0.8	-1.434712E-01	5.990267E-06

Example 3: Consider the linear boundary value problem

$$y^{(10)} - y'' + xy = (-8 + x - x^2)e^x, \quad 0 < x < 1 \tag{40}$$

subject to $y(0) = 1,\ y(1) = 0,\ y'(0) = 0,\ y'(1) = -e, \quad y''(0) = -1, \quad y''(1) = -2e,$

$$y'''(0) = -2, \quad y'''(1) = -3e, \ y^{(4)}(0) = -3, \ y^{(4)}(1) = -4e.$$

The exact solution for the above problem is $y = (1-x)e^x$.

The proposed method is tested on this problem where the domain [0, 1] is divided into 10 equal subintervals. The obtained numerical results for this problem are given in Table 3. The maximum absolute error obtained by the proposed method is 2.372265x10^{-5}.

Example 4: Consider the nonlinear boundary value problem

$$y^{(10)} + e^{-x}y^2 = e^{-x} + e^{-3x}, \qquad 0 < x < 1 \tag{41}$$

subject to $y(0) = 1, \ \ y(1) = e^{-1}, \ y'(0) = -1, \ y'(1) = -e^{-1}, \quad y''(0) = 1, \ \ y''(1) = e^{-1},$

$$y'''(0) = -1, \quad y'''(1) = -e^{-1}, \ y^{(4)}(0) = 1, \quad y^{(4)}(1) = e^{-1}.$$

The exact solution for the above problem is $y = e^{-x}$.

The nonlinear boundary value problem (41) is converted into a sequence of linear boundary value problems generated by quasilinearization technique [22] as

$$y_{(n+1)}^{(10)} + [2y_{(n)}e^{-x}]y_{(n+1)} = [y_{(n)}]^2 e^{-x} + e^{-x} + e^{-3x}, \qquad n = 0,1,2,3,\ldots \tag{42}$$

subject to $y_{(n+1)}(0) = 1, y_{(n+1)}(1) = e^{-1}, \ y'_{(n+1)}(0) = -1, y'_{(n+1)}(1) = -e^{-1}, \ y''_{(n+1)}(0) = 1, \ y''_{(n+1)}(1) = e^{-1},$

$$y'''_{(n+1)}(0) = -1, \ y'''_{(n+1)}(1) = -e^{-1}, \ y^{(4)}{}_{(n+1)}(0) = 1, \ y^{(4)}{}_{(n+1)}(1) = e^{-1}.$$

Here $y_{(n+1)}$ is the $(n+1)^{th}$ approximation for $y(x)$. The domain [0, 1] is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems (42) Numerical results for this problem are presented in Table 4. The maximum absolute error obtained by the proposed method is 1.138449x10^{-5}.

Table 3: Numerical Results for Example 3

x	Exact Solution	Absolute Error by Proposed Method
0.1	9.946538E-01	2.324581E-06
0.2	9.771222E-01	1.090765E-05
0.3	9.449012E-01	3.635883E-05
0.4	8.950948E-01	4.988909E-05

Table 3: Contd.,

0.5	8.243606E-01	4.547834E-05
0.6	7.288475E-01	3.725290E-05
0.7	6.041259E-01	1.931190E-05
0.8	4.451082E-01	8.672476E-06
0.9	2.459602E-01	7.867813E-06

Table 4: Numerical Results for Example 4

x	Exact Solution	Absolute Error by Proposed Method
0.1	9.048374E-01	1.788139E-07
0.2	8.187308E-01	5.304813E-06
0.3	7.408182E-01	5.960464E-06
0.4	6.703200E-01	9.417534E-06
0.5	6.065307E-01	1.138449E-05
0.6	5.488116E-01	2.086163E-06
0.7	4.965853E-01	1.043081E-06
0.8	4.493290E-01	8.344650E-07
0.9	4.065697E-01	2.235174E-06

Example 5: Consider the nonlinear boundary value problem

$$y^{(10)} - y''' = 2e^{x}y^{2}, \qquad 0 < x < 1 \tag{43}$$

subject to $y(0)=1,\ y(1)=e^{-1},\ y'(0)=-1,\ y'(1)=-e^{-1},\ \ y''(0)=1,\ y''(1)=e^{-1},$

$y'''(0)=-1,\ \ y'''(1)=-e^{-1},\ y^{(4)}(0)=1,\ \ y^{(4)}(1)=e^{-1}.$

The exact solution for the above problem is $y = e^{-x}$.

The nonlinear boundary value problem (43) is converted into a sequence of linear boundary value problems generated by quasilinearization technique [22] as

$$y^{(10)}_{(n+1)} - y''' - [4e^{x}y_{(n)}]y_{(n+1)} = -2[y_{(n)}]^{2}e^{x}, \qquad n = 0,1,2,3,\ldots \tag{44}$$

subject to $y_{(n+1)}(0)=1, y_{(n+1)}(1)=e^{-1},\ y'_{(n+1)}(0)=-1, y'_{(n+1)}(1)=-e^{-1},\ y''_{(n+1)}(0)=1,\ y''_{(n+1)}(1)=e^{-1},$

$y'''_{(n+1)}(0)=-1,\ \ y'''_{(n+1)}(1)=-e^{-1},\ \ y^{(4)}{}_{(n+1)}(0)=1,\ y^{(4)}{}_{(n+1)}(1)=e^{-1}.$

Here $y_{(n+1)}$ is the $(n+1)^{th}$ approximation for $y(x)$. The domain [0, 1] is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems (44). Numerical results for this problem are presented in Table 5. The maximum absolute error obtained by the proposed method is $1.138449\text{x}10^{-5}$.

Example 6: Consider the nonlinear boundary value problem

$$y^{(10)} = \frac{14175}{4}(x+y+1)^{11}, \qquad 0 \le x \le 1 \tag{45}$$

subject to $y(0)=1$, $y(1)=0$, $y'(0)=\dfrac{-1}{2},\ y'(1)=1,\ y''(0)=\dfrac{1}{2},\ y''(1)=4,\ y'''(0)=\dfrac{3}{4},\ \ y'''(1)=12,\ y^{(4)}(0)=\dfrac{3}{2},\ y^{(4)}(1)=48.$

The exact solution for the above problem is $y = \dfrac{2}{2-x} - x - 1.$

The nonlinear boundary value problem (45) is converted into a sequence of linear boundary value problems generated by quasilinearization technique [22] as

$$y_{(n+1)}^{(10)} - \frac{14175\times 11}{4}(x + y_{(n)} + 1)^{10}\, y_{(n+1)} = \frac{14175}{4}(x + y_{(n)} + 1)^{10}(1 + x - 10\, y_{(n)}),\ \ n = 0,1,2,\ldots \tag{46}$$

subject to $y_{(n+1)}(0)=1$, $y_{(n+1)}(1)=0$, $y'_{(n+1)}(0) = \dfrac{-1}{2}$, $y'_{(n+1)}(1) = 1$, $y''_{(n+1)}(0) = \dfrac{1}{2}$, $y''_{(n+1)}(1) = 4$,

$y'''_{(n+1)}(0) = \dfrac{3}{4}$, $y'''_{(n+1)}(1) = 12$, $y^{(4)}{}_{(n+1)}(0) = \dfrac{3}{2}$, $y^{(4)}{}_{(n+1)}(1) = 48.$

Here $y_{(n+1)}$ is the $(n+1)^{th}$ approximation for $y(x)$. The domain [0, 1] is divided into 10 equal subintervals and the proposed method is applied to the sequence of linear problems (46). Numerical results for this problem are presented in Table 6. The maximum absolute error obtained by the proposed method is 3.887713x10^{-5}.

Table 5: Numerical Results for Example 5

x	Exact Solution	Absolute Error by Proposed Method
0.1	9.048374E-01	1.788139E-07
0.2	8.187308E-01	5.304813E-06
0.3	7.408182E-01	5.960464E-06
0.4	6.703200E-01	9.417534E-06
0.5	6.065307E-01	1.138449E-05
0.6	5.488116E-01	2.086163E-06
0.7	4.965853E-01	1.043081E-06
0.8	4.493290E-01	8.344650E-07
0.9	4.065697E-01	2.235174E-06

Table 6: Numerical Results for Example 6

x	Exact Solution	Absolute Error by Proposed Method
0.1	-4.736842E-02	2.756715E-07
0.2	-8.888889E-02	3.091991E-06
0.3	-1.235294E-01	1.531094E-05
0.4	-1.500000E-01	3.047287E-05
0.5	-1.666667E-01	3.887713E-05
0.6	-1.714286E-01	3.656745E-05
0.7	-1.615385E-01	2.184510E-05
0.8	-1.333333E-01	8.359551E-06
0.9	-8.181816E-02	3.889203E-06

CONCLUSIONS

In this paper, we have deployed a Galerkin method with sextic B-splines as basis functions to solve a general tenth order boundary value problem. The sextic B-splines basis set has been redefined into a new set of basis functions which vanish on the boundary where the Dirichlet, Neumann, secondary order derivative, third order derivative and fourth order derivative types of boundary conditions are prescribed. The proposed method has been tested on three linear and three nonlinear tenth order boundary value problems. The numerical results obtained by the proposed method are in good agreement with the exact solutions available in the literature. The objective of this paper is to present a simple and accurate method to solve a general tenth order boundary value problem.

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